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Convergence to strong nonlinear diffusion waves for solutions to p -system with damping

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ABSTRACT

In this paper we consider the so-called p -system with linear damping, and we will prove an optimal decay estimates without any smallness conditions on the initial error. More precisely, if we restrict the initial data (V_0, U_0) in the space $H^3(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+) \times H^2(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+)$, then we can derive faster decay estimates than those given in [P. Marcati, M. Mei, B. Rubino, Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping, *J. Math. Fluid Mech.* 7 (2) (2005) 224–240; H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p -system with damping, *J. Differential Equations* 174 (1) (2001) 200–236] and [M. Jian, C. Zhu, Convergence to strong nonlinear diffusion waves for solutions to p -system with damping on quadrant, *J. Differential Equations* 246 (1) (2009) 50–77].

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1. Introduction

In this paper, we investigate the asymptotic stability of the following hyperbolic conservation laws with frictional damping of the form

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -\alpha u, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \quad (1.1)$$

We consider the following initial data

$$(v(t, x), u(t, x))|_{t=0} = (v_0(x), u_0(x)) \quad (1.2)$$

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which satisfies

$$(v_0(x), u_0(x)) \rightarrow (v_{\pm}, u_{\pm}) \quad \text{as } x \rightarrow \pm\infty, \quad (1.3)$$

and the null-Dirichlet boundary condition

$$u|_{x=0} = 0. \quad (1.4)$$

We recall that $v(t, x)$ and $u(t, x)$ represent the specific volume and velocity, respectively, whereas the function p represents the pressure where is assumed to be a smooth function of v such that $p(v) > 0$, $p'(v) < 0$ and α is a positive constant. (More specific assumptions on the function p will be given in the next section.) As it was well known in the literature [1,3,6,12–14,16,17], the system (1.1) can be viewed as the isentropic Euler equations in Lagrangian coordinates with the frictional term $-\alpha u$ in the momentum equation. Thus it models the compressible flow through porous media. Also system (1.1) can be used to model the motion of one-dimensional elastic continua interacting with media exerting frictional forces with deformation gradient strain v and stress $-p$.

The large time behavior of the p -system with damping was studied first by Hsiao and Liu in [3]. They showed that the solutions of (1.1) time-asymptotically behave like those governed by Darcy's law (2.1) in L^2 and L^∞ norms. Nishihara [13] succeeded in proving the following decay rates of convergence

$$\|(v - \bar{v}, u - \bar{u})\|_{L^\infty} = O(t^{-1}, t^{-3/2}),$$

when $v_+ = v_-$. The main tools used in [13] are the energy method and the pointwise estimates. After that the problem of the decay rates of solutions has been extensively studied in different cases. See for example [2,9,10,14,17] and references therein.

Nishihara et al. [16] showed the optimal decay rates for the Cauchy problem, which improves the works in [3] and [13], by using the method of pointwise estimates and the approximating Green function.

In [17] the authors proved that in the case of null-Dirichlet boundary condition on u , the solution (v, u) tends to $(v_+, 0)$ as t tends to infinity with the L^p rates $(1+t)^{-(\frac{3}{4}-\frac{1}{2p})}$ for $2 \leq p \leq \infty$, by using the Green function of the diffusion equation with constant coefficients.

The above L^p rates have been improved later by Marcati et al. [10] to be of the form $(1+t)^{-(1-\frac{1}{2p})}$, by using the energy method combined with the Fourier transform. In most of the previous works the smallness conditions on the initial error are necessary to obtain the global existence and the asymptotic stability.

In [19], Zhao proved that the initial error functions (V_0, U_0) can be chosen arbitrary large. More precisely, he proved the unique global solution of problem (1.1)–(1.4) and he showed that the obtained solution tends time-asymptotically to the solution of (2.1), where only the initial oscillations are required to be small. Also the result in [19] has been proved without any further smallness assumptions on the strength of the nonlinear diffusion wave.

Based on Zhao's paper, recently Jiang and Zhu succeed to prove the same results as in [10] without any smallness conditions on the initial error, their analysis based on some delicate energy estimates. The case of boundary type of the form

$$v|_{x=0} = g(t), \quad t \in \mathbb{R}^+, \quad (1.5)$$

where $g(t) \rightarrow v_+$ has been considered by Marcati and Mei [9]. They showed that the IBVP (1.1), (1.2) and (1.5) solutions converge time-asymptotically to the shifted nonlinear diffusion wave solutions of the Cauchy problem to the nonlinear parabolic equation given by the related Darcy's law (2.1). Other results concerning the boundary effects can be found in [4,7,17].

Also, the Cauchy problem of p -system with linear damping has been investigated by many researchers, we cite among others the following papers [8,15,18,20].

Lin and Zheng [8] analyzed the Cauchy problem to (1.1) and (1.2) when the pressure function $p(v)$ satisfies the γ -law for $0 \leq \gamma \leq 3$, that is to say $p(v) = k^2 v^{-\gamma}$. Under an appropriate condition on the initial data they showed that if $0 < \gamma \leq 1$ then the Cauchy problem (1.1)–(1.2) has a unique global C^1 -solution. Whereas for $1 \leq \gamma < 3$, the Cauchy problem (1.1)–(1.2) has C^1 -solution for only a finite time. By using the maximum principle, the result in [8] has been improved recently by Qiao and Zhu [18] by assuming a weaker conditions than those given in [8].

Zhu in [20] investigated the convergence rates to nonlinear diffusion waves for the weak entropy solutions of the Cauchy problem associated to (1.4) by the vanishing viscosity method together with the energy method.

In this paper we consider system (1.1)–(1.4), and we will prove an optimal decay estimates without any smallness conditions on the initial error. Let us describe the paper's content in more detail. To obtain the better estimates of our p -system (1.1)–(1.4), we will in the first step transform our problem to a second-order evolution equation (3.5) and obtain decay estimates of (3.5) by combining the arguments in [5] and [11]. With these estimates at hand, and by a nontrivial variant of the techniques in [10], we succeed to prove the optimal decay results to the solutions of (1.1)–(1.4). In fact, if we restrict the initial data in the space $H^3(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+) \times H^2(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+)$, then we can derive faster decay estimates than those in [10,19] and [6]. Note that, as in [6] and in contrast to [9,10,13,14,16,17], the initial error $\|V_0\|_3 + \|U_0\|_2$ has not considered to be small.

We point out that similar ideas can be used to study the hyperbolic conservation laws with nonlinear damping

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -\alpha u - \beta |u|^{q-1} u, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \quad (1.6)$$

where $\alpha > 0$ and $\beta \neq 0$. Such a problem will be considered in a forthcoming paper.

2. Notations and statement of the problem

Throughout this paper, $\|\cdot\|_q$ and $\|\cdot\|_{H^l}$ stand for the $L^q(\mathbb{R}^+)$ -norm ($2 \leq q \leq \infty$) and the $H^l(\mathbb{R}^+)$ -norm and sometimes for $L^q(\mathbb{R})$ -norm and the $H^l(\mathbb{R})$ -norm, respectively. Also, for $\gamma \in [0, +\infty)$, we define the weighted function space $L^{1,\gamma}(\mathbb{R}^+)$ as follows: $u \in L^{1,\gamma}(\mathbb{R}^+)$ iff $u \in L^1(\mathbb{R}^+)$ and

$$\|u\|_{1,\gamma} = \int_{\mathbb{R}^+} (1 + |x|)^\gamma |u(x)| dx < +\infty.$$

Similarly, we can define the space $L^{1,\gamma}(\mathbb{R})$.

Let us consider the system of nonlinear diffusion waves $(\bar{v}, \bar{u})(x, t)$ which satisfies the porous media equation

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ \bar{u}|_{x=0} = 0, \quad (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases} \quad (2.1)$$

System (2.1) is obtained by Darcy's law. By (2.1), (\bar{v}, \bar{u}) called the diffusion wave, is determined by

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ \bar{u} = -\frac{1}{\alpha} p(\bar{v})_x, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ \bar{u}|_{x=0} = 0, \quad (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases}$$

As in [3,10,16,17] and in order to eliminate the values of $u(x, t)$ at the end point $x = +\infty$ (or we assume that $u_+ = u_-$), we introduce a pair of auxiliary functions (\hat{u}, \hat{v}) as follows

$$\begin{cases} \hat{v}(x, t) = \frac{u_+ m_0(x)}{-\alpha} e^{-\alpha t}, \\ \hat{u}(x, t) = u_+ e^{-\alpha t} \int_0^x m_0(y) dy, \end{cases} \quad (2.2)$$

where $m_0(x)$ is a smooth function with compact support such that

$$\int_0^\infty m_0(y) dy = 1, \quad \text{supp } m_0 \subset \mathbb{R}^+, \quad m'_0(0) = 0.$$

It is not hard to verify that the pair (\hat{u}, \hat{v}) satisfies the following system

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, & x \in \mathbb{R}^+, t > 0, \\ \hat{u}_t = -\alpha \hat{u}, & \hat{u}|_{x=0} = 0, \quad (\hat{v}, \hat{u})|_{x=\infty} = (0, u_+ e^{-\alpha t}). \end{cases} \quad (2.3)$$

Similarly, as in [3], let us introduce the new variables

$$\begin{cases} V(x, t) = - \int_x^\infty (v - \bar{v} - \hat{v})(y, t) dy, \\ U(x, t) = u(x, t) - \bar{u}(x, t) - \hat{u}(x, t). \end{cases} \quad (2.4)$$

Combining the equations in (1.1), (2.1) and (2.3), we get

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x = -\alpha(u - \bar{u} - \hat{u}) - u_t. \end{cases} \quad (2.5)$$

Consequently, from (2.4) and (2.5), we obtain the new initial-boundary value problem

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v})V_x)_x + \alpha U = -F, & (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (V, U)|_{t=0} = (\bar{V}_0, \bar{U}_0)(x), \\ V|_{x=0} = 0, \end{cases} \quad (2.6)$$

where

$$F := \frac{1}{\alpha} p(\bar{v})_{xt} + (p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x)_x,$$

and

$$\begin{aligned} \bar{V}_0 &:= V_0(x) + \int_x^\infty \hat{v}(y, 0) dy = - \int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)) dy, \\ \bar{U}_0 &:= u_0(x) - \hat{u}(x, 0) = u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0). \end{aligned}$$

In order to state our main result, we assume that one of the following assumptions (P_1) or (P_2) holds:

(P_1) $p(v) \in C^3(0, \infty)$, $p'(v) < 0$, $p''(v) > 0$, $4p'(v)p'''(v) \geq 5(p''(v))^2$ for $v \in (0, \infty)$
and

$$\lim_{v \rightarrow 0} \int_v^1 \sqrt{-p'(\tau)} d\tau = +\infty, \quad \lim_{v \rightarrow \infty} p'(v) = 0,$$

or

(P_2) $p(v) \in C^2(\mathbb{R})$, $p'(v) < 0$ for $v \in \mathbb{R}$.

Under the above assumption, we can verify that system (1.1) has two eigenvalues

$$\lambda = -\sqrt{-p'(v)}, \quad \mu = \sqrt{-p'(v)} \quad (2.7)$$

and the corresponding Riemann invariants are taken as

$$r = u + h(v), \quad s = u - h(v), \quad (2.8)$$

where

$$h(v) = \int_a^v \mu(\tau) d\tau \quad (2.9)$$

with $a \in (0, \infty)$ is any fixed constant.

Remark 2.1. A typical example of a function p satisfying (P_1) is $p(v) = k^2 v^{-\gamma}$ where $1 \leq \gamma \leq 3$, this case has been considered in [8] and [18].

We state here the global existence result of [6].

Theorem 2.1. (See [6].) Under the assumption (P_1) (respectively (P_2)), and for arbitrarily given positive constants v_1, v_2 (respectively M_1) and M_2 , there exists a sufficiently small positive constant M_3 such that $(r_0(x), s_0(x)) \in C_b^1(\mathbb{R}^+)$ with

$$\begin{cases} v_1 \leq v_0(x) \leq v_2 & (\text{respectively } |v_0(x)| \leq M_1), \quad |u_0(x)| \leq M_2, \\ |r'_0(x)| \leq \alpha M_3, \quad |s'_0(x)| \leq \alpha M_3, \end{cases} \quad (2.10)$$

where

$$r_0(x) = u_0(x) + h(v_0(x)), \quad s_0(x) = u_0(x) - h(v_0(x)), \quad (2.11)$$

then the initial boundary value problem (1.1)–(1.4) admits a unique global in time smooth solution $(v(x, t), u(x, t))$.

3. The main result

Our main result and the novelty of this paper lies in the following theorem:

Theorem 3.1. *Under the hypothesis of Theorem 2.1, and in addition if $(V_0, U_0) \in H^3(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+) \times H^2(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+)$, then the solution $(v(x, t), u(x, t))$ given by Theorem 2.1 tends to the solution (\bar{u}, \bar{v}) of (2.1) and satisfies*

$$v - \bar{v} - \hat{v} \in C^k(0, \infty; H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$

$$u - \bar{u} - \hat{u} \in C^k(0, \infty; H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2.$$

Also, for $\gamma \in [0, 1]$, we have the following decay estimates

$$\begin{aligned} \|\partial_x^k(v - \bar{v} - \hat{v})\|_{L^2(\mathbb{R}^+)} &\leq O(1)(1+t)^{-\frac{(2k+3)}{4}-\frac{\gamma}{2}}, \quad k = 0, 1, \\ \|(v - \bar{v} - \hat{v})\|_{L^p(\mathbb{R}^+)} &\leq O(1)(1+t)^{-\frac{(2p-1)}{2p}-\frac{\gamma}{2}}, \quad 2 \leq p \leq +\infty, \\ \|(v - \bar{v} - \hat{v})\|_{L^2(\mathbb{R}^+)} &\leq O(1)(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}. \end{aligned} \quad (3.1)$$

Remark 3.1. We note here that the above result generalizes the existing results in the literature [9, 10, 12, 14, 16, 17, 19] in the sense that without any smallness assumptions on the initial error, we prove that our solution is global in time, and can decay with better rates than those proved in those papers. Also this result improves the decay rates given in the recent paper [6].

Concerning the solution of (2.6) and instead of the estimates (3.13) and (3.14) in [10], we have the following better estimates:

Theorem 3.2. *Under the assumption of Theorem 2.1, for $\gamma \in [0, 1]$, and $(V_0, U_0) \in H^3(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+) \times H^2(\mathbb{R}^+) \cap L^{1,\gamma}(\mathbb{R}^+)$, the solution $(V, U)(x, t)$ of (2.6) decays time-asymptotically as*

$$\begin{aligned} \|\partial_x^k V(t)\|_{L^2(\mathbb{R}^+)} &\leq O(1)(1+t)^{-\frac{(2k+1)}{4}-\frac{\gamma}{2}}, \quad k = 0, 1, 2, \\ \|U(t)\|_{L^2(\mathbb{R}^+)} &\leq O(1)(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}. \end{aligned} \quad (3.2)$$

Proof. The proof of Theorem 3.2 will be done based on the works in [11], [5] and [10]. Throughout this section we will denote by C a generic constant which may change from line to line.

As in [10], by setting $U = V_t$, we obtain from (2.6) the following initial and boundary value problem

$$\begin{cases} V_{tt} + \alpha V_t - \beta V_{xx} = G, & (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ (V, V_t)(x, 0) = (\bar{V}_0, \bar{U}_0)(x), & x \in \mathbb{R}^+, \\ V(0, t) = 0, \end{cases} \quad (3.3)$$

with $\beta = -p'(v_+)$ and $G = -F - ((p(\bar{v}) - p'(\bar{v}_+))V_x)_x$.

In our proof and in order to use the argument in [11], which is based on Fourier's transformation, we will extend our problem (3.3) on the whole domain \mathbb{R} . To do this we extend the solution V of (3.3) onto \mathbb{R} as odd function with respect to $x = 0$. Then we make the following odd extension

$$\begin{aligned} \psi(x, t) &:= \begin{cases} V(x, t), & x \geq 0, \\ -V(-x, t), & x < 0, \end{cases} \\ \psi_0 &:= \begin{cases} \bar{V}_0(x), & x \geq 0, \\ -\bar{V}_0(-x), & x < 0, \end{cases} \\ \psi_1 &:= \begin{cases} \bar{U}_0(x), & x \geq 0, \\ -\bar{U}_0(-x), & x < 0. \end{cases} \end{aligned} \quad (3.4)$$

Let us first take $G = 0$ in (3.3), then, similarly, as in [10], we get the following Cauchy problem with the new variable ψ

$$\begin{cases} \psi_{tt} + \alpha \psi_t - \beta \psi_{xx} = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\ (\psi, \psi_t)(x, 0) = (\psi_0, \psi_0')(x), & x \in \mathbb{R}. \end{cases} \quad (3.5)$$

Using the method in [11] (see also [10]), the solution of (3.5) can be represented as follows

$$\psi(x, t) = K_1 * \psi_1 + K_2 * \psi_0, \quad (3.6)$$

where the asterisk $*$ denotes the convolution, K_1 and K_2 can be represented as

$$K_j(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} R_j(\xi, t) d\xi, \quad j = 1, 2.$$

Here $R_j(t, \xi)$ be the Fourier transform of $K_j(t, x)$ ($j = 1, 2$). Thus, R_j satisfies the following ODE systems

$$\begin{aligned} \frac{d^2}{dt^2} R_j + \alpha \frac{d}{dt} R_j + \beta |\xi|^2 R_j &= 0, \quad j = 1, 2, \\ R_1(0, \xi) &= 0, \quad \frac{d}{dt} R_1(0, \xi) = 1, \\ R_2(0, \xi) &= 1, \quad \frac{d}{dt} R_2(0, \xi) = 0. \end{aligned} \quad (3.7)$$

As in [11], by solving the above systems, we get the following results:

$$R_1(t, \xi) = \begin{cases} \frac{2e^{-\alpha t/2}}{\sqrt{\alpha^2 - 4\beta|\xi|^2}} \sinh\left(\frac{\sqrt{\alpha^2 - 4\beta|\xi|^2}}{2} t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}}, \\ te^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}}, \\ \frac{2e^{-\alpha t/2}}{\sqrt{4\beta|\xi|^2 - \alpha^2}} \sin\left(\frac{\sqrt{4\beta|\xi|^2 - \alpha^2}}{2} t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}}, \end{cases} \quad (3.8)$$

and

$$R_2(t, \xi) = \frac{\alpha}{2} R_1(t, \xi) + R_3(t, \xi),$$

where

$$R_3(t, \xi) = \begin{cases} e^{-\alpha t/2} \cosh\left(\frac{\sqrt{\alpha^2 - 4\beta|\xi|^2}}{2} t\right), & |\xi| < \frac{\alpha}{2\sqrt{\beta}}, \\ e^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\beta}}, \\ \cos\left(\frac{\sqrt{4\beta|\xi|^2 - \alpha^2}}{2} t\right), & |\xi| > \frac{\alpha}{2\sqrt{\beta}}. \end{cases} \quad (3.9)$$

Furthermore, due to Eq. (3.6), it follows that

$$\|\psi\|_2 \leq C(\|K_1 * \psi_1\|_2 + \|K_2 * \psi_0\|_2). \quad (3.10)$$

It is well known that

$$\|f\|_{L^k} \leq \|\mathcal{F}(f)\|_{L^m}, \quad \frac{1}{k} + \frac{1}{m} = 1, \quad 1 \leq m \leq 2, \quad (3.11)$$

where $\mathcal{F}(f)$ denotes the Fourier transform of f .

By using the above inequality, (3.10) becomes

$$\begin{aligned} \|\psi\|_2 &\leq C(\|\mathcal{F}(K_1(t, \cdot)(\xi))\mathcal{F}(\psi_1)\|_2 + \|\mathcal{F}(K_2(t, \cdot)(\xi))\mathcal{F}(\psi_0)\|_2) \\ &\leq C(\|R_1(t, \cdot)(\xi)\mathcal{F}(\psi_1)\|_2 + \|R_2(t, \cdot)(\xi)\mathcal{F}(\psi_0)\|_2). \end{aligned} \quad (3.12)$$

The following lemma is the key element of the proof of our result, it is quite similar to Lemma 1.1 in [11].

Lemma 3.1. *Let $\gamma \in [0, 1]$. If $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k-1}(\mathbb{R})$, then*

$$\|\partial_t^j \partial_x^k (K_1 * f)\|_2 \leq C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} [\|f\|_{L^{1,\gamma}(\mathbb{R})} + \|f\|_{H^{j+k-1}(\mathbb{R})}]. \quad (3.13)$$

If $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k}(\mathbb{R})$, then

$$\|\partial_t^j \partial_x^k (K_2 * f)\|_2 \leq C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} [\|f\|_{L^{1,\gamma}(\mathbb{R})} + \|f\|_{H^{j+k}(\mathbb{R})}]. \quad (3.14)$$

Proof. In order to prove the above lemma, we will use the method in [11] together with the idea in the work of Ikehata [5], but here the presence of the high derivatives make the proof very delicate.

By taking into account the inequality (3.12), it suffices to derive the decay estimates of the terms of the right-hand side of that inequality for the odd function $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k-1}(\mathbb{R})$ and $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k}(\mathbb{R})$, respectively.

To make the proof easy we take (only in the proof of this lemma) $\alpha = \beta = 1$. We can deal with the general case by making the following change of variables $T = \alpha t$ and $x = \frac{\sqrt{\beta}}{\alpha} \xi$.

By using (3.11), we get

$$\|\partial_t^j \partial_x^k (K_1 * f)\|_2^2 \leq \left\| (i\xi)^k \frac{d^j}{dt^j} R_1(t, \xi) \cdot \mathcal{F}(f(\xi)) \right\|_2^2 = \int_{\mathbb{R}} |\xi|^{2k} \left| \frac{d^j}{dt^j} R_1(t, \xi) \cdot \mathcal{F}(f(\xi)) \right|^2 d\xi. \quad (3.15)$$

Now, as in [11], we fix a small $0 < \delta < 1/2$, and we divide the integral in the right-hand side of (3.15) into four main parts as follows

$$\int_{\mathbb{R}} |\xi|^{2k} \left| \frac{d^j}{dt^j} R_1(t, \xi) \cdot \mathcal{F}(f(\xi)) \right|^2 d\xi = \int_{|\xi| \geq 1} + \int_{1/2 \leq |\xi| \leq 1} + \int_{\delta \leq |\xi| \leq 1/2} + \int_{|\xi| \leq \delta}. \quad (3.16)$$

The first three integral terms in (3.16) can be estimated exactly as in [11], consequently, we have

$$\begin{aligned} I_1 &\leq C e^{-t} \|f\|_{H^{j+k-1}}^2, \\ I_2 &\leq C(1+t) e^{-t} \|f\|_2^2, \\ I_3 &\leq C e^{-(1-\sqrt{1-4\delta^2})t} \|f\|_2^2. \end{aligned} \quad (3.17)$$

Now, we will use the trick of [5] to proof a crucial estimate of the integral term I_4 .

Consequently, we have the following result.

Lemma 3.2. *Let us suppose that $\gamma \in [0, 1]$. If $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k-1}(\mathbb{R})$ is an odd function with respect to $x = 0$, then the following estimate holds*

$$I_4 \leq C_\gamma (1+t)^{-(k+2j+\gamma)-1/2} \|f\|_{L^{1,\gamma}(\mathbb{R})}^2. \quad (3.18)$$

The proof of Lemma 3.2 will be done later.

Consequently, using (3.16), (3.17) and (3.18), we have the desired estimate (3.13).

The proof of (3.14) is quite similar as the one of (3.13). Indeed, similar to (3.15), we have

$$\begin{aligned} \|\partial_t^j \partial_x^k (K_2 * f)\|_2^2 &\leq \left\| (i\xi)^k \frac{d^j}{dt^j} R_2(t, \xi) \cdot \mathcal{F}(f(\xi)) \right\|_2^2 \\ &= \int_{\mathbb{R}} |\xi|^{2k} \left| \frac{d^j}{dt^j} (R_1(t, \xi) + R_3(t, \xi)) \cdot \mathcal{F}(f(\xi)) \right|^2 d\xi. \end{aligned} \quad (3.19)$$

By taking into account the estimate (3.13), it suffices to estimate the following integral

$$\begin{aligned} \int_{\mathbb{R}} |\xi|^{2k} \left| \frac{d^j}{dt^j} (R_3(t, \xi)) \cdot \mathcal{F}(f(\xi)) \right|^2 d\xi &= \int_{|\xi| \geq 1} + \int_{1/2 \leq |\xi| \leq 1} + \int_{\delta \leq |\xi| \leq 1/2} + \int_{|\xi| \leq \delta} \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned} \quad (3.20)$$

Compared with the proof given by Matsumura [11] there is a little changes in the proof of the estimate of K_1 . Indeed, it is clear that

$$\begin{aligned} K_1 &= \int_{|\xi| \geq 1} |\xi|^{2k} \left| \frac{d^j}{dt^j} (R_3(t, \xi)) \cdot \mathcal{F}(f(\xi)) \right|^2 d\xi \\ &\leq C e^{-t} \int_{|\xi| \geq 1} |\xi|^{2k} \left(\frac{\sqrt{4|\xi|^2 - 1}}{2} + 1 \right)^{2j} |\mathcal{F}(f(\xi))|^2 d\xi \\ &\leq C e^{-t} \sup_{|\xi| \geq 1} \frac{(\sqrt{4|\xi|^2 - 1} + 1)^{2j}}{|\xi|^{2j}} \int_{|\xi| \geq 1} |\xi|^{2k+2j} |\mathcal{F}(f(\xi))|^2 d\xi. \end{aligned} \quad (3.21)$$

Therefore, (3.21) implies that

$$K_1 \leq C e^{-t} \|f\|_{H^{j+k}}^2. \quad (3.22)$$

The estimates of K_2 and K_3 are similar to those of I_2 and I_3 , respectively. Thus

$$\begin{aligned} K_2 &\leq C(1+t)e^{-t} \|f\|_2^2, \\ K_3 &\leq C e^{-(1-\sqrt{1-4\delta^2})t} \|f\|_2^2. \end{aligned} \quad (3.23)$$

Concerning the integral K_4 , we have the following result.

Lemma 3.3. *Let us suppose that $\gamma \in [0, 1]$. If $f \in L^{1,\gamma}(\mathbb{R}) \cap H^{j+k}(\mathbb{R})$ is an odd function with respect to $x = 0$, then the following estimate holds*

$$K_4 \leq C_\gamma (1+t)^{-(k+2j+\gamma)-1/2} \|f\|_{L^{1,\gamma}(\mathbb{R})}^2. \quad (3.24)$$

Consequently, from (3.13), (3.19), (3.20), (3.22), (3.23) and (3.24), we get that (3.14) holds. This completes the proof of Lemma 3.1. \square

Proof of Lemma 3.2. From (3.16) we have

$$I_4 = \int_{|\xi| \leq \delta} |\xi|^{2k} \left| \frac{d^j}{dt^j} R_1(t, \xi) \mathcal{F}(f(\xi)) \right|^2 d\xi. \quad (3.25)$$

As in [5], and since f is an odd function, we get

$$\mathcal{F}(f(\xi)) = -2i \int_0^\infty f(x) \sin(x\xi) d\xi.$$

Consequently, it is clear that

$$|\mathcal{F}(f(\xi))| \leq 2 \int_0^\infty |f(x)| \sin(x\xi) d\xi. \quad (3.26)$$

Now, it is clear that

$$\begin{aligned} \left(e^{-t/2} \sinh\left(\frac{\sqrt{1-4|\xi|^2}}{2} t\right) \right)^2 &= \frac{1}{4} (e^{-t(1-\sqrt{1-4|\xi|^2})/2} - e^{-t(1+\sqrt{1-4|\xi|^2})/2})^2 \\ &\leq e^{-t(1-\sqrt{1-4|\xi|^2})}. \end{aligned}$$

Consequently, by using the definition of R_1 , we obtain

$$|I_4| \leq C \int_{-\delta}^{\delta} \frac{(1 - \sqrt{1-4|\xi|^2})^{2j} |\xi|^{2k}}{\sqrt{1-4|\xi|^2}} |\mathcal{F}(f(\xi))|^2 \times e^{-t(1-\sqrt{1-4|\xi|^2})} d\xi.$$

Since $-4|\xi|^2 \leq -1 + \sqrt{1-4|\xi|^2} \leq -2|\xi|^2$, for $|\xi| \leq \delta < \frac{1}{2}$, then the above inequality takes the form

$$\begin{aligned} |I_4| &\leq C \int_{-\delta}^{\delta} |\xi|^{2k+4j} e^{-2t|\xi|^2} |\mathcal{F}(f(\xi))|^2 \\ &= C \int_{-\delta}^0 |\xi|^{2k+4j} e^{-2t|\xi|^2} |\mathcal{F}(f(\xi))|^2 + \int_0^{\delta} |\xi|^{2k+4j} e^{-2t|\xi|^2} |\mathcal{F}(f(\xi))|^2 \\ &= C(I_{4-} + I_{4+}). \end{aligned} \quad (3.27)$$

We will estimate I_{4+} by using the method in [5], the same arguments work for I_{4-} , we omit the details. Indeed

$$I_{4+} = \int_0^{\delta} |\xi|^{2k+4j} e^{-2t|\xi|^2} |\mathcal{F}(f(\xi))|^2 d\xi.$$

The inequality (3.26) implies

$$I_{4+} \leq 4 \int_0^\delta |\xi|^{2k+4j} e^{-2t|\xi|^2} \left(\int_0^\infty |f(x)| |\sin(x\xi)| \right)^2 dx. \quad (3.28)$$

Let us fix $\varepsilon > 0$, then for each $\xi > 0$, we obtain

$$\begin{aligned} \int_\varepsilon^\infty |f(x)| |\sin(x\xi)| dx &= \int_\varepsilon^\infty (x\xi)^\gamma |f(x)| \frac{|\sin(x\xi)|}{(x\xi)^\gamma} dx \\ &\leq \xi^\gamma \int_\varepsilon^\infty (1+x)^\gamma |f(x)| M_\gamma dx \\ &\leq M_\gamma \xi^\gamma \int_\varepsilon^\infty (1+x)^\gamma |f(x)| dx, \end{aligned} \quad (3.29)$$

where

$$M_\gamma = \sup_{\theta > 0} \frac{|\sin \theta|}{\theta^\gamma}$$

is a constant independent of ε . It is clear that $M_\gamma < +\infty$ since $\gamma \in [0, 1]$.

Once (3.29) holds for any $\varepsilon > 0$, then letting ε tend to 0, (3.29) implies

$$\int_0^\infty |f(x)| |\sin(x\xi)| dx \leq M_\gamma \xi^\gamma \|f\|_{1,\gamma}. \quad (3.30)$$

Consequently, for any $\varepsilon > 0$ (3.28) gives

$$\begin{aligned} \int_\varepsilon^\delta |\xi|^{2k+4j} e^{-2t|\xi|^2} \left(\int_0^\infty |f(x)| |\sin(x\xi)| \right)^2 dx \\ \leq M_\gamma^2 \|f\|_{1,\gamma}^2 \int_0^\delta |\xi|^{2k+4j+2\gamma} e^{-2t|\xi|^2} d\xi. \end{aligned} \quad (3.31)$$

Similarly, letting $\varepsilon \rightarrow 0$ once again, we conclude

$$I_{4+} \leq M_\gamma \xi^\gamma \|f\|_{1,\gamma}^2 \int_0^\delta |\xi|^{2k+4j+2\gamma} e^{-2t|\xi|^2} d\xi. \quad (3.32)$$

By exploiting the following inequality

$$\int_0^\delta |\xi|^\sigma e^{-2t|\xi|^2} d\xi \leq C(1+t)^{-(\sigma+1)/2}, \quad (3.33)$$

we deduce

$$\int_0^\delta |\xi|^{2k+4j+2\gamma} e^{-2t|\xi|^2} d\xi \leq C(1+t)^{-(k+2j+\gamma)-1/2}. \quad (3.34)$$

Consequently, we have

$$I_{4+} \leq C_\gamma (1+t)^{-(k+2j+\gamma)-1/2} \|f\|_{1,\gamma}^2. \quad (3.35)$$

By carrying the same calculation of I_{4-} , our desired result holds. This completes the proof of Lemma 3.2. \square

Proof of Lemma 3.3. From (3.26), (3.29) and with the same methods as in the estimation of I_4 , it is not hard to check that for $|\xi| \leq 1/2$, we have

$$\left| \frac{d^j}{dt^j} R_3(t, \xi) \right|^2 \leq C |\xi|^{4j} e^{-2t|\xi|^2}.$$

Consequently, we deduce

$$\begin{aligned} K_4 &\leq C \int_{|\xi| \leq \delta} |\xi|^{2k+4j} |\mathcal{F}(f(\xi))|^2 e^{-2t|\xi|^2} d\xi \\ &\leq CM_\gamma^2 \|f\|_{1,\gamma}^2 \int_{|\xi| \leq \delta} |\xi|^{2k+2\gamma+4j} e^{-2t|\xi|^2} d\xi. \end{aligned}$$

Therefore, (3.24) holds by a simple application of (3.33). \square

The following lemma will play a decisive role in the proof of our results. This lemma improves Lemma 4.2 in [10].

Lemma 3.4. Let $\gamma \in [0, 1]$. If $f \in L^{1,\gamma}(\mathbb{R}_+) \cap H^{j+k-1}(\mathbb{R}_+)$, then

$$\begin{aligned} &\left\| \partial_t^j \partial_x^k \int_0^\infty [K_1(x-y, t) - K_1(x+y, t)] f(y) dy \right\|_{L^2(\mathbb{R}_+)} \\ &\leq C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} [\|f\|_{L^{1,\gamma}(\mathbb{R}_+)} + \|f\|_{H^{j+k-1}(\mathbb{R}_+)}]. \end{aligned} \quad (3.36)$$

If $f \in L^{1,\gamma}(\mathbb{R}_+) \cap H^{j+k}(\mathbb{R}_+)$, then

$$\begin{aligned} &\left\| \partial_t^j \partial_x^k \int_0^\infty [K_2(x-y, t) - K_2(x+y, t)] f(y) dy \right\|_{L^2(\mathbb{R}_+)} \\ &\leq C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} [\|f\|_{L^{1,\gamma}(\mathbb{R}_+)} + \|f\|_{H^{j+k}(\mathbb{R}_+)}]. \end{aligned} \quad (3.37)$$

The proof of the above lemma lies on the method used in [10], but we must use Lemma 3.1 instead of Matsumura's Lemma used in [10]. So we omit its details here.

In order to finish the proof of Theorem 3.2, using the Duhamel's principle (see [10] for details), we can represent the solution $V(t, x)$ of (3.3) as follows

$$\begin{aligned} V(x, t) = & \int_0^\infty [K_2(x-y, t) - K_2(x+y, t)] \bar{V}_0(y) dy \\ & + \int_0^\infty [K_1(x-y, t) - K_1(x+y, t)] \bar{U}_0(y) dy \\ & + \int_0^t \int_0^\infty [K_1(x-y, t-\tau) - K_1(x+y, t-\tau)] G(y, \tau) dy d\tau. \end{aligned} \quad (3.38)$$

Consequently, the remaining part of the proof can be finished, following the same steps as in [10], we omit the details. \square

Proof of Theorem 3.1. We recall that the first part of Theorem 3.1 has been proved in [6, Theorem 2.4]. Then in order to prove Theorem 3.1, we only need to check the estimates (3.1).

We recall that the solution of (2.3) satisfies the following decay estimate (see [6, Lemma 2.1]):

$$\|\partial_x^k \partial_t^j \hat{v}(\cdot, t)\|_{L^p(\mathbb{R}_+)} \leq O(1) e^{-\alpha t}. \quad (3.39)$$

Since $V_x = v - \bar{v} - \hat{v}$, $U = u - \bar{u} - \hat{u}$, then by exploiting (3.2) and (3.38), we find

$$\begin{aligned} \|\partial_x^k (v - \bar{v} - \hat{v})\|_{L^2(\mathbb{R}_+)} &= \|\partial_x^k V_x\|_{L^2(\mathbb{R}_+)} \\ &\leq O(1) (1+t)^{-\frac{(2k+1)}{4} - \frac{\gamma}{2}} \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} \|(u - \bar{u} - \hat{u})\|_{L^2(\mathbb{R}_+)} &= \|U\|_{L^2(\mathbb{R}_+)} \\ &\leq O(1) (1+t)^{-\frac{5}{4} - \frac{\gamma}{2}}. \end{aligned} \quad (3.41)$$

This gives us exactly the first and the third estimates in (3.1). For the proof of the second estimate in (3.1), we use the first estimate in (3.1) and the Sobolev inequalities

$$\|f\|_{L^\infty} \leq \sqrt{2} \|f\|_{L^2}^{1/2} \|f_x\|_{L^2}^{1/2}$$

and

$$\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-2)/p} \|f\|_{L^2}^{2/p}$$

for $2 \leq p \leq +\infty$. Then we find

$$\begin{aligned}
\|(v - \bar{v} - \hat{v})\|_{L^p} &\leq \|(v - \bar{v} - \hat{v})\|_{L^\infty}^{(p-2)/p} \|(v - \bar{v} - \hat{v})\|_{L^2}^{2/p} \\
&\leq (\sqrt{2} \|(v - \bar{v} - \hat{v})\|_{L^2}^{1/2} \|\partial_x(v - \bar{v} - \hat{v})\|_{L^2}^{1/2})^{(p-2)/p} \|(v - \bar{v} - \hat{v})\|_{L^2}^{2/p} \\
&\leq O(1)(1+t)^{-(\frac{3}{4} + \frac{\gamma}{2})\frac{p-2}{2p}} (1+t)^{-(\frac{5}{4} + \frac{\gamma}{2})\frac{p-2}{2p}} \\
&\leq O(1)(1+t)^{-\frac{(2p-1)}{2p} - \frac{\gamma}{2}}.
\end{aligned} \tag{3.42}$$

Thus, the proof of Theorem 3.1 is completed. \square

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